The fundamental solution for small steady three-dimensional disturbances to a two-dimensional parallel shear flow

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CONTENTS

1. INTRODUCTION

In addition to the enormous aerodynamical literature concerned with the disturbances to a uniform stream produced by the presence of obstacles, there is a very extensive literature on disturbances to irrotational non-uniform streams. The smaller body of work on disturbances to rotational streams is concerned principally with two-dimensional disturbances to a parallel shear flow; these are fairly easy to treat because the vortex lines are straight; they remain parallel and do not get stretched at all during the motion.

By contrast, in flows where the disturbances are three-dimensional in character (even though the undisturbed stream may still be two-dimensional), stretching and rotation of the vortex lines must play an important part. In spite of the difficulties which this introduces, certain outstanding papers on the theory of these flows have appeared.

Squire (1933) found the equation governing small three-dimensional disturbances to a two-dimensional parallel flow. If the velocity field is

$$(V(y)+u, v, w), \tag{1}$$

then by neglecting squares of the disturbance velocities u, v, w and analysing them into Fourier components, he showed that disturbances varying with x and z like $e^{i(\alpha x+\beta z)}$ must vary with time like $e^{-i\alpha ct}$, where c is such that the equation

$$(V-c)\{v''-(\alpha^2+\beta^2)v\}-V''v-(\nu/i\alpha)\{v^{\mathrm{iv}}-2(\alpha^2+\beta^2)v''+(\alpha^2+\beta^2)^2v\}=0 \quad (2)$$

(where ν is kinematic viscosity and where primes signify differentiation with respect to y) is satisfied (together with boundary conditions) by a non-vanishing value of the y-velocity v. Values of c with negative imaginary part correspond to stable disturbances, values with positive imaginary part to unstable disturbances, and real values to neutral disturbances. Equation (2) shows that neutral disturbances with $\beta=\beta_0\neq 0$, $\alpha=\alpha_0$, $\nu=\nu_0$ can exist, if and only if neutral disturbances with

$$\beta=0, \qquad \alpha=\sqrt{(\alpha_0^2+\beta_0^2)}, \qquad \nu=\nu_0\sqrt{(1+\beta_0^2/\alpha_0^2)}$$

exist; the latter are two-dimensional disturbances at a lower Reynolds number*. Squire deduced the classical theorem that two-dimensional disturbances become unstable at a lower Reynolds number than three-dimensional disturbances.

Kármán & Tsien (1945) investigated small disturbances to a three-dimensional parallel stream (so that V in (1) is a function of z as well as of y) but limited themselves (as we shall do in this paper) to steady disturbances with viscosity neglected, evolving the theory of the 'lifting line' in such a non-uniform stream. They found it convenient to work from the equation for the pressure p, namely

$$\mathrm{div}\left(\frac{\mathrm{grad}\,p}{V^2}\right)=0, \qquad (3)$$

both because it in this case is simpler than that for any of the velocity components, and because the boundary condition in the lifting-line problem is expressed most conveniently in terms of p. The results of the lifting-line theory are complicated, however, and only in the case when V is a function of y alone (the spanwise coordinate) do they become reasonably tractable.

Another approach to the study of steady three-dimensional disturbances to a parallel shear flow has been to regard them as a 'secondary flow', to be analysed by the method initiated by Squire & Winter (1951). In this method there is no assumption that the disturbances are small; for example, the disturbance due to a sphere can be treated (Hawthorne & Martin 1955; Lighthill 1956). There is, however, an assumption that the undisturbed stream is weakly sheared. The 'primary flow' (or first approximation to the real flow) is taken to coincide (either wholly, or at least as regards its streamline pattern) with that in which the undisturbed stream is uniform. The secondary flow is a perturbation on this, due to allowing a small shear in the undisturbed stream. This shear is usually taken to be uniform, although such an assumption is not essential to the method; we shall call secondary flows calculated on this assumption 'simple-shear secondary flows'. The secondary vorticity can be calculated either direct from

* Equation (2) with $\beta=0$ is the Orr–Sommerfeld equation.

Helmholtz's equations (Hawthorne 1951, 1954), or by geometrical considerations (Preston 1954; Lighthill 1956) from the effect of the primary flow in stretching and rotating the vortex lines of the undisturbed shear flow. (From this point of view, the approximation adopted consists in neglecting the fact that it is really the exact flow, not the approximate 'primary flow', which carries the vortex lines along with it.) The calculation of the secondary velocity field from the associated vorticity field is more arduous, but can be achieved in special cases (Lighthill 1956, 1957b).

The work of this paper originated from a desire to clear up a difficulty arising in the application of the secondary-flow method to cases where the undisturbed stream is unbounded. The difficulty is that the secondary-flow disturbance due to the presence of an obstacle falls off more slowly with distance from it than does the primary-flow disturbance. For example, a 'half-body' extending from the origin to infinity downstream has a primary-flow disturbance of 'source' type, falling off like r^{-2} (where r is distance from the origin), but the corresponding secondary-flow disturbance falls off like r^{-1}. If the procedure of successive approximation were continued, and a tertiary-flow field calculated from the configuration of vortex lines as distorted by the sum of the primary and secondary flows, this would not even tend to zero as $r \to \infty$.

As another example, a finite body makes a primary-flow disturbance of 'doublet' type, falling off like r^{-3}, but the corresponding secondary-flow disturbance falls off like r^{-2} (see Lighthill (1956), where also a not quite accurate attempt was made to calculate the tertiary flow, but the conclusion that it falls off like r^{-1} was right).

Now, there is no experimental evidence (e.g. from the flow visualizations reported by Livesey (1956)) that secondary flows are large far from the obstacle, and it seems clear that, as suggested in Lighthill (1956), the approximation sequence is not uniformly valid in this region, where (therefore) the first term gives no indication of the real behaviour of the disturbance.

There is some analogy in this to the difficulties concerning flow about an obstacle at very low Reynolds numbers, often described as 'Stokes's and Whitehead's paradoxes'. In those problems a direct procedure of expansion in powers of Reynolds number gives nonsensical results, at any rate far from the obstacle (although for three-dimensional obstacles the first term in the expansion has value near the obstacle). The difficulty was resolved by Oseen, who showed that the correct form of the equations of motion at large distances is that obtained by neglecting squares of disturbances to the uniform stream but not discarding terms of lower order in the Reynolds number.

Similarly in the present problem the true behaviour of the flow far from the body is found from the full equations with the squares of the disturbances neglected. Such an approach is obviously valid sufficiently far from the obstacle, and the only problem is how the results obtained should be related to secondary-flow and other results obtained by the methods described above.

The answer found below is that the small-disturbance solution is valid in a region, far from the obstacle, which overlaps the region, near the obstacle, where the small-shear solution (more accurately, the 'simple-shear secondary flow') is valid. This overlapping holds in the sense that the asymptotic behaviour of the small-disturbance solution as one goes near the obstacle is identical with the asymptotic behaviour of the small-shear solution as one goes away from the obstacle. The overlap makes it possible to say how the flow behaves everywhere.

These are good reasons for studying small steady three-dimensional disturbances to a parallel shear flow. Obviously, however, there are many others: to investigate, for example, the flow of sheared winds over hills. Again, one could follow up the work of Kármán & Tsien (1945) by studying further kinds of perturbation of parallel shear flows by thin shapes of aerodynamic interest.

As a basic tool for any such investigation, we seek in this paper the 'fundamental solution' of the equation. This represents the flow due to a source (but a weak source, since our equation treats only small disturbances) in a parallel shear flow. This solution will give as it stands the asymptotic behaviour of the disturbances due to a 'half-body', since these are equivalent far from the body to those produced by a source. Also, a simple differentiation of the solution will give the disturbance due to a doublet, which will represent the asymptotic behaviour of the disturbances due to a finite body. Finally, more complicated solutions of the equations can be built up (on familiar lines) once the fundamental solutions for a source and a doublet are known; work on these lines, however, is excluded from the present paper.

We limit ourselves (like Squire 1933) to the case when the undisturbed flow is two-dimensional, its velocity field being $(V(y), 0, 0)$. The formal theory for the undisturbed velocity field $(V(y, z), 0, 0)$ (used by Kármán & Tsien 1945) can be derived, but so much extra effort is then needed to deduce any concrete results (as indeed they also found) that it is better to try to keep the theory intelligible by confining discussion to the simpler case. In this case the equation, after Fourier transforms with respect to x and z have been taken, becomes Squire's equation (2) with $c = 0$ (because the disturbances are steady). There are, of course, no continuous solutions of this equation which satisfy the appropriate boundary conditions (such solutions exist only for one value of c, which in practice is non-zero); on the other hand, to represent a source (say, at the origin), one needs a solution with a discontinuity of a certain specified kind at $y = 0$. This solution is the Fourier transform of the required disturbance field, which must be deduced from it by Fourier's inversion theorem.

The Orr–Sommerfeld equation has in the past been used almost exclusively for studying harmonic disturbances. The building up of anharmonic disturbance fields by Fourier synthesis from solutions of that equation, as in this paper and the author's previous paper on boundary layers and upstream influence in supersonic flow (Lighthill 1953), is an approach that seems fruitful and may have a number of applications.

In the present paper the viscous terms in the equation are neglected. This is known to be permissible except where $V(y)$ is near the 'critical' values 0 and c, which coincide in the case here studied. Thus, shear layers in which the undisturbed velocity falls to zero anywhere are excluded in this paper as a result of the neglect of the viscous terms; such shear layers include boundary layers and jets. Boundary layers could be treated, provided that the viscous terms in the equation were retained in some 'inner viscous layer' near the wall, as used by Lighthill (1953) to provide a corrected form of boundary condition for the non-viscous equation. However, in a preliminary survey this refinement was thought to be not worth making, particularly since laminar boundary layers when disturbed are so prone to large disturbances associated with separation. With jets, again, the assumption of parallel or even nearly parallel flow at the edge, where air is being entrained, is wrong. Accordingly, the theory as set out here is limited to a wake, or a mixing region between parallel streams, or any other layer in which $V(y)$ is nowhere zero.

Since steady inviscid flow is being considered the Kármán–Tsien equation (3) could have been used. However, equation (2) for v (with $\nu = c = 0$), although no simpler than equation (3) for p, has advantages in the present problem, in which v (but not p) turns out to be a function of y and of $s = \sqrt{(x^2+z^2)}$ alone, and in which the velocity components, rather than the pressure, are the main things one wants to know.

Of the results of the theory, four main classes may be mentioned in advance. First, if the solution is expanded in ascending powers of r (distance from the source), the first term (of order r^{-2}) is the primary flow (the source flow itself) and the second (of order r^{-1}) is the small-disturbance approximation to the simple-shear secondary flow (as already mentioned) and depends only on $V(0)$ and $V'(0)$, the velocity and shear at the origin. However, the next term (which might be called the small-disturbance form of the 'tertiary flow') depends on the velocity distribution throughout the shear layer.

Another result of general interest is concerned with the behaviour of the solution for large r: a source in a shear layer produces in any region of uniform flow outside the shear layer a disturbance equivalent to a source of different strength in a different position. The strength of the equivalent source can alternatively be predicted by simpler arguments (due to Mr M. B. Glauert), in which the shear layer is regarded as a discontinuity between regions of uniform flow. The displacement in effective position is of the order of the layer width, and has not yet been predicted by such simpler arguments.

A third result is that in shear layers with small velocity spread (that is, a velocity minimum only a few per cent less than the velocity maximum) a specially simple form of solution is available; this can be given an elegant interpretation by means of images, again due to Mr M. B. Glauert. It can also be shown to be identical with the asymptotic form of the 'exact-profile secondary flow', that is, of the secondary flow calculated by allowing the vortex lines of the exact undisturbed-flow profile $V(y)$ (not the corresponding simple-shear profile $V(0)+V'(0)y$) to be stretched and rotated by the primary flow.

Lastly, the theory is applied to the problem of the displacement of the dividing streamline in shear flows about axisymmetrical obstacles—a problem which the author has used in a number of recent papers, both as a convenient excuse for developing techniques for calculating shear flows, and as a point where theory could easily be referred to the results of experiments. In the last section of this paper it is shown what corrections need to be made, to values of this displacement as calculated from simple-shear secondary-flow theory, to take into account the improved picture of what the disturbances are like far from the obstacle.

2. Equations of motion

The momentum equation for steady incompressible inviscid flow is

$$\mathbf{v}.\nabla\mathbf{v}+\frac{1}{\rho}\nabla p = 0, \tag{4}$$

where $\mathbf{v}$ is the velocity, ρ the density and p the pressure. If the departure from two-dimensional parallel flow is small, so that

$$\mathbf{v} = (V(y)+u, v, w), \tag{5}$$

where squares and products of u, v, w and their derivatives are to be neglected, equation (4) becomes

$$V(y)\frac{\partial u}{\partial x}+vV'(y)+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{6}$$

$$V(y)\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{7}$$

$$V(y)\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z} = 0. \tag{8}$$

We shall assume that all disturbances vanish far upstream (at $x = -\infty$), so that (7) and (8), with the pressure eliminated, give

$$\frac{\partial}{\partial y}\{V(y)w\} = -\frac{1}{\rho}\int_{-\infty}^{x}\frac{\partial^2 p}{\partial y\partial z}dx = V(y)\frac{\partial v}{\partial z}. \tag{9}$$

Elimination of p from (6) and (7) gives, similarly,

$$\frac{\partial}{\partial y}\left\{V(y)\frac{\partial u}{\partial x}\right\} = V(y)\frac{\partial^2 v}{\partial x^2}-\frac{\partial}{\partial y}\{V'(y)v\}. \tag{10}$$

These equations are to be satisfied together with the equation of continuity. Now, to obtain the 'fundamental solution', due to a source of strength m at the origin, from which other solutions can be built up, we write that equation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x} = m\delta(x)\delta(y)\delta(z), \tag{11}$$

where $\delta(x)$ is the delta function of Dirac, or 'impulse function'; thus, the right-hand side of (11) vanishes except at the origin, but its integral over any volume including the origin is m. No similar term appears on the right-hand side of (6), (7) or (8) because the source is taken to be a source of mass but not of momentum.

To obtain an equation for the single variable v, we first multiply (11) by $V(y)$, giving

$$V(y)\frac{\partial u}{\partial x} + V(y)\frac{\partial v}{\partial y} + V(y)\frac{\partial w}{\partial z} = mV(0)\delta(x)\delta(y)\delta(z) \tag{12}$$

(where the general theorem $f(y)\delta(y) = f(0)\delta(y)$ has been used), and then we differentiate (12) with respect to y, using (9) and (10), to obtain

$$V(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial y}\{V'(y)v\} + \frac{\partial}{\partial y}\left\{V(y)\frac{\partial v}{\partial y}\right\} + V(y)\frac{\partial^2 v}{\partial z^2} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{13}$$

or

$$V(y)\nabla^2 v - V''(y)v = mV(0)\delta(x)\delta'(y)\delta(z). \tag{14}$$

Equation (14) for v is the basic equation of motion. After it has been solved, w and u may easily be deduced, from (9) in the form

$$V(y)w = \int_{\pm\infty}^{y} V(y)\frac{\partial v}{\partial z}dy, \tag{15}$$

and from the equation of continuity (11). The choice of lower limit in (15) arises from the condition that the disturbance vanish at $y = \pm\infty$, but either limit gives the same answer. This is because

$$\int_{-\infty}^{\infty} V(y)v\,dy = 0. \tag{16}$$

To prove (16) (which is a useful check on the accuracy of solutions, and has been applied as such to all those given in this paper), integrate (13) from $y = -\infty$ to $y = +\infty$. This shows that the left-hand side of (16) is a harmonic function of the two variables x and z for all values of those variables; but it vanishes at infinity, and so must be zero everywhere.

To obtain u, on the other hand, one must integrate the equation of continuity with respect to x from $-\infty$ (not $+\infty$) to x, since it is possible that fluid at $x = +\infty$, whose vorticity may have been permanently redistributed when it passed the source, may have non-zero disturbance velocity.

It is possible that the solution of the basic equation (14), for given $V(y)$, could be obtained by some sort of 'analogue method'. Thus, the equation can be written

$$\operatorname{div}\left\{V^2(y)\operatorname{grad}\frac{v}{V(y)}\right\} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{17}$$

which shows that $v/V(y)$ is proportional to the electrostatic potential of a dipole at the origin, with axis the y-axis, in a stratified medium with dielectric constant proportional to $V^2(y)$. Many similar analogies are, of course, possible, and some might, perhaps, be useful in giving intuitive information about the solution, or an experimental procedure for computing it. The author has not so far been able to make such use of them, and in the meantime offers an analytical method of solution.

It may also be noted that the theory can be at once extended to problems with density stratification, in which the undisturbed flow has its density $\rho(y)$ as well as its velocity $(V(y), 0, 0)$ dependent on y; however, this extension is straightforward only if gravity is still negligible—thus, the variable inertia of the fluid, but not the variable weight, can easily be taken into account. The equation which results in this case, corresponding to (17), is

$$\operatorname{div}\left\{V^2(y)\frac{\rho(y)}{\rho(0)}\operatorname{grad}\frac{v}{V(y)}\right\} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{18}$$

so that the expression for $v/V(y)$ may be derived from its expression for constant ρ by simply replacing $V(y)$ in the latter expression by

$$V(y)\surd\{\rho(y)/\rho(0)\}.$$

Unfortunately this result is not of much value because in problems with density stratification the variable buoyancy of the fluid is usually important (or else the Mach number is not negligible). One may hope that the theory can later be extended to take the gravitational term into account, but this would require a new investigation.

To solve equation (14) it is usually convenient to divide it by $V(y)$, giving

$$\nabla^2 v - \frac{V''(y)}{V(y)}v = m\delta(x)\delta(z)\left\{\delta'(y) + \frac{V'(0)}{V(0)}\delta(y)\right\}, \tag{19}$$

where the form of the right-hand side follows from the general theorem

$$f(y)\delta'(y) = f(0)\delta'(y) - f'(0)\delta(y).$$

One next writes v as a Fourier integral

$$v(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta z)}v_0(y, \alpha, \beta)\, d\alpha d\beta. \tag{20}$$

Then, by substitution in (19),

$$v_0'' - \left(\alpha^2 + \beta^2 + \frac{V''}{V}\right)v_0 = \frac{m}{4\pi^2}\left\{\delta'(y) + \frac{V'(0)}{V(0)}\delta(y)\right\}, \tag{21}$$

where all primes signify differentiation with respect to y.

Apart from the delta functions on the right-hand side, representing the effect of the source, equation (21) is the same as Squire's equation (2), with c and ν put equal to 0. The fact that α and β appear only in the combination

$$\alpha^2 + \beta^2 = k^2 \tag{22}$$

is again important. It follows from it that v itself depends only on y and on

$$x^2 + z^2 = s^2. \tag{23}$$

For if we substitute

$$\alpha = k\cos\theta, \qquad \beta = k\sin\theta, \qquad x = s\cos\lambda, \qquad z = s\sin\lambda \tag{24}$$

in (20), then, since $d\alpha d\beta = k\,dk d\theta$, we obtain

$$v = \int_0^\infty v_0 k\,dk \int_0^{2\pi} e^{iks\cos(\theta-\lambda)}\,d\theta$$

$$= 2\pi \int_0^\infty v_0 k J_0(ks)\,dk. \tag{25}$$

Thus v, a function of y and s alone, is the 'Hankel transform' (25) of v_0, a function of y and k alone.

This v_0 is to be obtained from (21), which states that for $y > 0$ and $y < 0$ the simple equation

$$v_0'' - v_0\left(k^2 + \frac{V''}{V}\right) = 0 \tag{26}$$

is satisfied by v_0, but that v_0 and v_0' are discontinuous at $y = 0$, in such a way that

$$v_0(+0) - v_0(-0) = \frac{m}{4\pi^2}, \qquad v_0'(+0) - v_0'(-0) = \frac{m}{4\pi^2}\frac{V'(0)}{V(0)}. \tag{27}$$

The problem has thus been reduced to terms not involving any generalized functions.

3. Simple solutions for two particular cases

There are two particular, although somewhat artificial, distributions $V(y)$ of the undisturbed velocity for which very simple solutions exist. First, if

$$V(y) = U + Ay, \tag{28}$$

so that the oncoming flow is uniformly sheared, then (26) becomes

$$v_0'' - k^2 v_0 = 0, \tag{29}$$

and the solution of (29) which tends to zero as $|y| \to \infty$ and satisfies (27) is

$$v_0 = \frac{m}{8\pi^2} e^{-k|y|}\operatorname{sgn} y - \frac{m}{8\pi^2 k}\frac{V'(0)}{V(0)} e^{-k|y|}, \tag{30}$$

where $\operatorname{sgn} y$ is $+1$ when $y > 0$ and -1 when $y < 0$. By the result (Watson 1944, § 13.2 (1))

$$\int_0^\infty e^{-k|y|} J_0(ks)\,dk = \frac{1}{\sqrt{(y^2+s^2)}}, \tag{31}$$

and its derivative with respect to y, equations (30) and (25) give

$$v = \frac{\partial}{\partial y}\left(-\frac{m}{4\pi r}\right) - \frac{V'(0)}{V(0)}\frac{m}{4\pi r}, \tag{32}$$

where $r = \sqrt{(y^2+s^2)} = \sqrt{(x^2+y^2+z^2)}$. That the coefficient of $V'(0)/V(0)$ must be simply the potential of a source of strength m is independently obvious from equation (19) when the second term therein vanishes; similarly, the other term must be the y-derivative of the source potential $(-m/4\pi r)$.

The first term in (32) is the 'primary' flow due to the source, and the second term is the small-disturbance approximation to the simple-shear secondary flow (Lighthill 1957a). The full secondary flow in the simple-shear case (28) is the solution obtained by assuming not small source strength but small shear; it is a perturbation on the irrotational flow due to the source, derived by neglecting the square of the shear; at a large distance from the source, where the disturbance to the oncoming stream is also small, it takes the form shown in (32). What is interesting in (32) is that no higher powers of the shear (representing tertiary, quartary flows, etc.) appear, although in the present analysis these have not been neglected. It is, as we shall see, only for the particular, and artificial, undisturbed velocity distribution (28) that this is so.

Moreover, such terms are not absent in w. By (32) and (15), with $V = U+Ay$,

$$w = \frac{1}{U+Ay}\int_{-\infty}^{y}(U+Ay)\left(\frac{\partial}{\partial y}+\frac{A}{U}\right)\left(\frac{mz}{4\pi r^3}\right)dy, \tag{33}$$

which after an integration by parts gives

$$w = \frac{mz}{4\pi r^3} - \frac{A^2}{U(U+Ay)}\frac{mz}{4\pi r}, \tag{34}$$

of which the last term is a tertiary-flow term. More seriously, this term makes the integration of the equation of continuity to obtain u impossible without allowing u to become logarithmically infinite at both $x = +\infty$ and $x = -\infty$. This is an extremely discouraging feature of the present solution. However, it is related to the circumstance that $V(y)$ actually becomes zero for finite y, and we shall see that for more realistic velocity distributions the difficulty does not arise.

There is one other $V(y)$, not specially realistic, but at least not leading to unrealistic infinities, for which a simple solution is possible. This is

$$V(y) = Ue^{\lambda y}. \tag{35}$$

Again, the solution is most readily obtained from equation (19), which in this case becomes

$$\nabla^2 v - \lambda^2 v = \left(\frac{\partial}{\partial y}+\lambda\right)m\delta(x)\delta(y)\delta(z), \tag{36}$$

whence

$$v = \left(\frac{\partial}{\partial y}+\lambda\right)\left[-\frac{me^{-\lambda r}}{4\pi r}\right], \tag{37}$$

the term in square brackets being the fundamental solution of $\nabla^2 v = \lambda^2 v$. Alternatively, (37) can be obtained by first solving (26) and then making use of the result (Watson 1944, § 13.47 (2) with $\mu = 0$, $\nu = \frac{1}{2}$)

$$\int_0^\infty \frac{k}{\sqrt{(k^2+\lambda^2)}}e^{-|y|\sqrt{(k^2+\lambda^2)}}J_0(ks)\,dk = \frac{e^{-\lambda\sqrt{(y^2+s^2)}}}{\sqrt{(y^2+s^2)}}. \tag{38}$$

The modification of the primary- and secondary-flow terms in (37) by the exponential factor inside the square bracket means that tertiary-flow terms in λ^2 and higher-order terms are present in v in this case, but that they all add up to a solution tending to zero rapidly as $r \to \infty$. As to w, equation (15) with (37) gives simply

$$w = \frac{\partial}{\partial z}\left(-\frac{me^{-\lambda r}}{4\pi r}\right), \tag{39}$$

and for u the equation of continuity yields after some reduction

$$u = \frac{\partial}{\partial x}\left(-\frac{me^{-\lambda r}}{4\pi r}\right) + \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)\int_{-\infty}^{x} \frac{e^{-\lambda r}}{r}\,dx. \tag{40}$$

Thus there is an exponential falling-off as $r \to \infty$ in u also, except as $x \to +\infty$ for values of y and z which are not large. This last result is realistic, and corresponds to the fact that fluid which has passed close to the source has been permanently affected, because its vortex lines have been deformed. In fact,

$$\begin{aligned}\lim_{x\to+\infty}(u) &= \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)\int_{-\infty}^{\infty} \frac{e^{-\lambda\sqrt{(x^2+y^2+z^2)}}}{\sqrt{(x^2+y^2+z^2)}}\,dx \\ &= \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)2K_0\{\lambda\sqrt{(y^2+z^2)}\},\end{aligned} \tag{41}$$

by Watson (1944), § 6.22 (5). All the outflow due to the source has somehow been channelled into the region immediately downstream of it*. That the total volume flow in the x-direction in this region is in fact m is easily verified from (41); we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\lim_{x\to+\infty} u)\,dy\,dz = m\int_0^{\infty} tK_0(t)\,dt = m, \tag{42}$$

by Watson (1944), § 13.21 (8). The solution, however, is not worth pursuing further because of the impossibility that in a real flow the velocity distribution could be well represented by (35) over any large range of y.

4. Method of solution in the general case

We pass to methods applicable for a general distribution of velocity $V(y)$, subject only to the restriction that the limits $V(\infty)$ and $V(-\infty)$ exist, and that $V(y)$ is nowhere zero, even at these limits. (We shall also require $V(y)$ to tend to these limits sufficiently fast for certain integrals to converge.) Thus, the parallel flow is supposed to become uniform as $y \to +\infty$ and as $y \to -\infty$, as in a wake or in a mixing region between two streams.

The equation (26) as $y \to +\infty$ tends to the simple form (29) and has solutions e^{-ky} and e^{ky}. Only the former tends to zero and so is of interest to us. We therefore introduce the notation $v_1(y, k)$ for the solution of (26) which is asymptotic to e^{-ky} as $y \to +\infty$. Similarly, $v_2(y, k)$ signifies the

* We shall see that this is due to the displacement of streamlines, as they pass the source, towards the region of lower undisturbed velocity.

solution which is asymptotic to e^{ky} as $y \to -\infty$. The necessary discontinuous solution v_0 must take the form

$$v_0 = A(k)v_1(y,k) \quad (y > 0), \qquad B(k)v_2(y,k) \quad (y < 0), \tag{43}$$

for some constants A and B. But, by (27),

$$A(k)v_1(0,k) - B(k)v_2(0,k) = \frac{m}{4\pi^2}, \tag{44}$$

$$A(k)v_1'(0,k) - B(k)v_2'(0,k) = \frac{m}{4\pi^2}\frac{V'(0)}{V(0)}, \tag{45}$$

and the solution of these simultaneous equations is

$$\left.\begin{aligned} A(k) &= \frac{m}{4\pi^2 W(k)}\left\{v_2'(0,k) - \frac{V'(0)}{V(0)}v_2(0,k)\right\}, \\ B(k) &= \frac{m}{4\pi^2 W(k)}\left\{v_1'(0,k) - \frac{V'(0)}{V(0)}v_1(0,k)\right\}, \end{aligned}\right\} \tag{46}$$

where

$$W(k) = v_1(y,k)v_2'(y,k) - v_2(y,k)v_1'(y,k) \tag{47}$$

is independent of y by a well-known property of second-order linear differential equations (and it is actually (47) for $y = 0$ that has been used to derive (46)).

Equation (46) would be useless if the Wronskian $W(k)$ vanished. However, this cannot happen. For it would mean that v_1 and v_2 were simply proportional to one another, so that a *continuous* (and continuously differentiable) solution v_0 existed tending to zero at $y = \pm\infty$. Physically, a stationary wave could then be present without the need for any extraneous disturbance like the source here discussed. This state of affairs is not at all plausible; but the possibility can be dismissed altogether on mathematical grounds, since if such a v_0 existed, it may be shown from the differential equation (26) that

$$\int_{-\infty}^{\infty}\left\{\left(v_0' - \frac{V'}{V}v_0\right)^2 + k^2 v_0^2\right\} dy = 0, \tag{48}$$

so that v_0 must vanish identically (because otherwise the left-hand side would be positive).

The character of the solution $v_1(y,k)$ (and its sister solution $v_2(y,k)$), on which all has been made to depend, will be sufficiently uncovered in what follows by three theories, one for large k, one for small k, and one for general k subject to the restriction that the total variation of $V(y)$ is small compared with its absolute magnitude. In each case the corresponding properties of $v_0(y,k)$, $v(y,s)$, u and w are inferred from the above formulas and those of § 2.

5. Theory for small r

By properties of the Hankel transformation (25) we shall be able to deduce the behaviour of v for small r from the behaviour of v_0 for large k.

To study the latter we solve equation (26) in the form

$$v_0'' - k^2 v_0 = \frac{V''}{V}v_0$$

by 'variation of parameters'. Picking the solution v_1 asymptotic to e^{-ky} as $y \to +\infty$, we obtain

$$v_1(y,k) = e^{-ky}\left\{1 + \int_y^\infty \frac{1-e^{-2k(q-y)}}{2k}\frac{V''(q)}{V(q)} v_1(q,k)e^{kq}\,dq\right\}. \tag{49}$$

This integral equation is useful for many purposes, in particular for proving by Picard's well-known method the existence and uniqueness of $v_1(y,k)$, and its analyticity as a function of k for $\mathscr{R}\{k\} \geqslant 0$. It also tells us the expansion of $v_1(y,k)$ for large k, giving to a first approximation $v_1 e^{ky} = 1 + O(k^{-1})$ and to a second

$$v_1 = e^{-ky}\left\{1 + \frac{1}{2k}\int_y^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k^2}\right)\right\}. \tag{50}$$

Similarly, $$v_2 = e^{ky}\left\{1 + \frac{1}{2k}\int_{-\infty}^y \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k^2}\right)\right\}, \tag{51}$$

and so the Wronskian of v_1 and v_2 is

$$W(k) = v_1 v_2' - v_2 v_1' = 2k + \int_{-\infty}^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k}\right), \tag{52}$$

its independence of y being a check on the work. Hence, by (46),

$$\begin{aligned} A &= \frac{m}{4\pi^2}\left\{2k + \int_{-\infty}^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k}\right)\right\}^{-1} \times \\ &\qquad \times \left\{k + \frac{1}{2}\int_{-\infty}^0 \frac{V''(q)}{V(q)}\,dq - \frac{V'(0)}{V(0)} + O\left(\frac{1}{k}\right)\right\} \\ &= \frac{m}{8\pi^2}\left\{1 - \frac{1}{2k}\int_0^\infty \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}, \end{aligned} \tag{53}$$

and similarly

$$B = \frac{m}{8\pi^2}\left\{-1 + \frac{1}{2k}\int_{-\infty}^0 \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}, \tag{54}$$

whence by (43), (50) and (51)

$$v_0 = \frac{m}{8\pi^2}e^{-k|y|}\left\{\operatorname{sgn} y - \frac{1}{2k}\int_0^y \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}. \tag{55}$$

The author has evaluated also the coefficient of k^{-2} in the expansion of $v_0(y,k)$ for large k, but the only property of this coefficient that tells us anything about the behaviour of v for small r is that it is $O(|y|)$ as $|y| \to 0$; the next term still is one in k^{-3}. Hence, writing $F(y,k)$ for the error term in (55), we have by (25) and (31)

$$\begin{aligned} v &= \frac{my}{4\pi r^3} - \left\{\frac{V'(0)}{V(0)} + \frac{1}{2}\int_0^y \frac{V''(q)}{V(q)}\,dq\right\}\frac{m}{4\pi r} + \\ &\qquad + \frac{m}{4\pi}\int_0^\infty e^{-k|y|}J_0(ks)F(y,k)\,dk, \end{aligned} \tag{56}$$

with $$|F(y,k)| < \frac{C_1|y|}{1+k} + \frac{C_2}{(1+k)^2}$$

for some C_1 and C_2. The integral in (56) is therefore bounded as $r \to 0$; it is less in modulus than

$$C_1|y| \int_0^\infty \frac{e^{-k|y|}\,dk}{1+k} + C_2 \int_0^\infty \frac{dk}{(1+k)^2}$$

which is bounded, because the first integral is $O(\log|y|^{-1})$. (Actually, the integral in (56) tends to a finite limit $\int_0^\infty F(0,k)\,dk$ as $r \to 0$. However, this limit cannot be evaluated unless $v_0(0,k)$ is known for all k.) Part of the middle term in (56) is also bounded, and so finally

$$v = \frac{my}{4\pi r^3} - \frac{V'(0)}{V(0)} \frac{m}{4\pi r} + O(1) \quad \text{as } r \to 0. \tag{57}$$

Equation (57) shows that the part of v which tends to infinity as $r \to 0$ consists simply of the primary flow and the simple-shear secondary flow. In other words, near the source the variations of velocity and of shear are small enough for the approximations of neglecting the square of the shear and regarding it as uniform to give the right answer.

It follows from (57) and (15) that

$$w = \frac{\partial}{\partial z}\left(-\frac{m}{4\pi r}\right) + O(1); \tag{58}$$

this also is the sum of the primary flow and the simple-shear secondary flow, since the latter has no z-component if the squares of the disturbances are neglected. Finally, the equation of continuity gives

$$u = \frac{\partial}{\partial x}\left(-\frac{m}{4\pi r}\right) - \frac{m}{4\pi} \frac{V'(0)}{V(0)} \frac{y}{y^2+z^2}\left(1+\frac{x}{r}\right) + O(1), \tag{59}$$

the second term being obtained by integrating y/r^3 with respect to x from the lower limit $-\infty$; again, this second term agrees with the small-disturbance approximation to the simple-shear secondary flow (Lighthill 1957a).

The work of this section gives a check that fluid is truly emerging from the source at the rate m units of volume per unit time, as well as showing up the character of the simple-shear secondary flow solution as a valid second approximation *in the neighbourhood of the source.*

6. Evaluation of v_0 for small k

By a process inverse to that of § 5, the behaviour of v for large r can be deduced from that of v_0 for small k, which is obtained as follows.

A uniformly valid first approximation to the solution $v_1(y,k)$ for small k is

$$v_1 = \frac{V(y)}{V(\infty)} e^{-ky}. \tag{60}$$

To see this, note that when y is large $V(y) \to V(\infty)$, so (60) tends to e^{-ky} as it should, but when y is not large $e^{-ky} \doteqdot 1$ (for small k), and so (60) is approximately proportional to $V(y)$, which in turn approximately satisfies (26) for small k. These facts suggest the transformation

$$v_1 = \frac{V(y)}{V(\infty)} e^{-ky} u_1, \tag{61}$$

which on substitution in (26) gives the equation

$$\frac{d}{dy}\left(Y e^{-2ky} \frac{du_1}{dy}\right) = kY' e^{-2ky} u_1, \tag{62}$$

where for shortness we have written $V^2(y)$ as $Y(y)$. From (62) and the condition $u_1 \to 1$ as $y \to \infty$ we obtain the integral equation

$$u_1(y) = 1 + k \int_y^\infty \frac{dq}{Y(q)} \int_q^\infty Y'(l) e^{2k(q-l)} u_1(l)\, dl, \tag{63}$$

whence successive approximations to u_1 for small k may be found, the first being $u_1 = 1 + O(k)$, the second

$$u_1 = 1 + k \int_y^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq + O(k^2), \tag{64}$$

and the third

$$u_1 = 1 + k \int_y^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq - k^2 \int_y^\infty \frac{dq}{Y(q)} \times \\ \times \int_q^\infty \frac{\{Y(\infty) - Y(l)\}\{Y(q) + Y(l)\}}{Y(l)}\, dl + O(k^3). \tag{65}$$

Similarly, we put

$$v_2 = \frac{V(y)}{V(-\infty)} e^{ky} u_2, \tag{66}$$

and find that

$$u_2 = 1 + k \int_{-\infty}^y \frac{Y(-\infty) - Y(q)}{Y(q)}\, dq - \\ - k^2 \int_{-\infty}^y \frac{dq}{Y(q)} \int_{-\infty}^q \frac{\{Y(-\infty) - Y(l)\}\{Y(q) + Y(l)\}}{Y(l)}\, dl + O(k^3). \tag{67}$$

Since also

$$u_1' = -k \frac{Y(\infty) - Y(y)}{Y(y)} + \frac{k^2}{Y(y)} \int_y^\infty \frac{\{Y(\infty) - Y(q)\}\{Y(y) + Y(q)\}}{Y(q)}\, dq + O(k^3), \tag{68}$$

and

$$u_2' = k \frac{Y(-\infty) - Y(y)}{Y(y)} - \frac{k^2}{Y(y)} \int_{-\infty}^y \frac{\{Y(-\infty) - Y(q)\}\{Y(y) + Y(q)\}}{V(q)} \times \\ \times dq + O(k^3), \tag{69}$$

we infer that the Wronskian is

$$W(k) = v_1 v_2' - v_2 v_1' = \frac{Y(y)}{V(\infty) V(-\infty)} (u_1 u_2' - u_2 u_1' - 2k u_1 u_2) \\ = \frac{k\{Y(\infty) + Y(-\infty)\}}{V(\infty) V(-\infty)} + \frac{k^2}{V(\infty) V(-\infty)} \times \\ \times \int_{-\infty}^\infty \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{Y(q)}\, dq + O(k^3), \tag{70}$$

where on collecting terms it was found that all those involving y explicitly disappear, which is a check on the result.

Now, by (46) and (66),

$$A = \frac{m}{4\pi^2 W(k)} \frac{V(0)}{V(-\infty)} \{u_2'(0,k) + ku_2(0,k)\}$$

$$= \frac{m}{4\pi^2 W(k)} \frac{V(0)}{V(-\infty)} \left[k\frac{Y(-\infty)}{Y(0)} - \frac{k^2}{Y(0)} \times \right.$$

$$\left. \times \int_{-\infty}^{0} \{Y(-\infty) - Y(q)\}\, dq + O(k^3) \right], \quad (71)$$

and similarly

$$B = \frac{m}{4\pi^2 W(k)} \frac{V(0)}{Y(\infty)} \left[-k\frac{(Y\infty)}{Y(0)} + \frac{k^2}{Y(0)} \int_0^\infty \{Y(\infty) - Y(q)\}\, dq + O(k^3) \right], \quad (72)$$

results which with equation (70) give non-zero limits for A and B as $k \to 0$ and give also their derivatives with respect to k at $k = 0$.

By (43), (61), (65), (70) and (71), for $y > 0$,

$$v_0 = \frac{mV(y)e^{-ky}}{4\pi^2 V(0)} \frac{Y(-\infty)}{Y(\infty) + Y(-\infty)} \times$$

$$\times \left[1 + k\int_y^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq - k\int_{-\infty}^{0} \frac{Y(-\infty) - Y(q)}{Y(-\infty)}\, dq - \right.$$

$$\left. -k\int_{-\infty}^{\infty} \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{\{Y(\infty) + Y(-\infty)\}Y(q)}\, dq + O(k^2) \right], \quad (73)$$

and similarly for $y < 0$

$$v_0 = -\frac{mV(y)e^{ky}}{4\pi^2 V(0)} \frac{Y(\infty)}{Y(\infty) + Y(-\infty)} \times$$

$$\times \left[1 + k\int_{-\infty}^{y} \frac{Y(-\infty) - Y(q)}{Y(q)}\, dq - k\int_0^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq - \right.$$

$$\left. -k\int_{-\infty}^{\infty} \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{\{Y(\infty) + Y(-\infty)\}Y(q)}\, dq + O(k^2) \right]. \quad (74)$$

Now, equations (73) and (74) are to be used differently in different regions. For large $|y|$, they are best used as they stand, or at most after such simplification as is gained from replacing $V(y)$ by $V(\infty)$ or $V(-\infty)$, as the case may be, and suppressing the first integral inside each square bracket. However, for values of $|y|$ which are not large, it is permissible, and valuable, to expand the $e^{-k|y|}$ factor outside the bracket in each equation and take it inside. When this is done, both (73) and (74) can be rearranged into a common form in which the coefficient of k is seen to be a continuously differentiable function of y even at $y = 0$. This form for $|y|$ not large is

$$v_0 = \frac{mV(y)}{4\pi^2 V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)} \left[\frac{\operatorname{sgn} y}{Y(\infty \operatorname{sgn} y)} - k \int_0^y \frac{dq}{Y(q)} + \right.$$
$$+ \frac{k}{Y(\infty)+Y(-\infty)} \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)} dq - \right.$$
$$\left. \left. - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)} dq \right\} + O(k^2) \right]. \quad (75)$$

The term independent of k in (75) is still discontinuous at $y = 0$, but this will be found to produce no discontinuity in v itself, which is influenced only by the term proportional to k.

7. Theory for large r

We now show that the properties of

$$v = 2\pi \int_0^\infty v_0(y,k) k J_0(ks)\, dk \quad (25\,\text{bis})$$

for large $r = \sqrt{(y^2+s^2)}$ can be deduced from the equations (73), (74) and (75) which give the behaviour of $v_0(y,k)$ for small k.

The case when $|y|$ is itself large is especially easy and may as well be treated first. For large positive y, (73) shows that

$$v_0 \sim \frac{m_1}{8\pi^2} e^{-ky}(1-kc_1) \quad (76)$$

as $k \to 0$, where

$$m_1 = \frac{V(\infty)}{V(0)} \frac{2V^2(-\infty)}{V^2(\infty)+V^2(-\infty)} m, \quad (77)$$

and

$$c_1 = \int_{-\infty}^0 \frac{Y(-\infty)-Y(q)}{Y(-\infty)} dq + \int_{-\infty}^\infty \frac{\{Y(\infty)-Y(q)\}\{Y(-\infty)-Y(q)\}}{\{Y(\infty)+Y(-\infty)\}Y(q)} dq. \quad (78)$$

The 'Borel' theory of integrals with exponential factors whose exponent consists of the variable of integration multiplied by a large parameter (here y) tells us that as $y \to \infty$

$$v_1 \sim \frac{1}{4\pi} \int_0^\infty m_1 e^{-ky}(1-kc_1) J_0(ks)\, dk = -\left(m_1 \frac{\partial}{\partial y} + m_1 c_1 \frac{\partial^2}{\partial y^2} \right) \frac{1}{4\pi r}, \quad (79)$$

where (31) has been used. The behaviour of w follows at once from (15) with the lower limit replaced by $+\infty$, in which V can be regarded as constant on both sides, giving

$$w \sim -\left(m_1 \frac{\partial}{\partial z} + m_1 c_1 \frac{\partial^2}{\partial y \partial z} \right) \frac{1}{4\pi r}. \quad (80)$$

The equation of continuity then gives a similar result for u, and the three results may be written

$$(u, v, w) \sim \operatorname{grad}\left(-\frac{m_1}{4\pi r}\right)+c_1\frac{\partial}{\partial y}\operatorname{grad}\left(-\frac{m_1}{4\pi r}\right). \tag{81}$$

Equation (81) shows that, for large positive y, the effect of the source of strength m at the origin, in the midst of the sheared flow, is the same as if a source of different strength m_1, and a doublet of strength $m_1 c_1$ with its axis in the y-direction, were present at the origin, but immersed in a uniform flow with velocity $V(\infty)$. (These are the two leading terms in the expansion of the irrotational disturbance above the shear layer in terms of singularities at the origin.)

Alternatively, we can say that the effect is the same as if a simple source of strength m_1 were present, not at the origin, but at the displaced position $(0, -c_1, 0)$.

Similarly, for large negative y,

$$(u, v, w) \sim \operatorname{grad}\left(-\frac{m_2}{4\pi r}\right)-c_2\frac{\partial}{\partial y}\operatorname{grad}\left(-\frac{m_2}{4\pi r}\right), \tag{82}$$

where

$$m_2 = \frac{V(-\infty)}{V(0)}\frac{2V^2(\infty)}{V^2(\infty)+V^2(-\infty)}m, \tag{83}$$

and

$$c_2 = \int_0^\infty \frac{Y(\infty)-Y(q)}{Y(q)}dq+\int_{-\infty}^\infty \frac{\{Y(\infty)-Y(q)\}\{Y(-\infty)-Y(q)\}}{\{Y(\infty)+Y(-\infty)\}Y(q)}dq. \tag{84}$$

Thus, for large negative y, the disturbance is that which would be produced in a uniform flow by a source of yet different strength m_2 at the origin, and a doublet of strength $m_2 c_2$ with its axis in the negative y-direction. Again, it can if preferred be regarded as equivalent to a source of strength m_2 at the displaced position $(0, +c_2, 0)$.

The strengths m_1 and m_2 of the equivalent sources can alternatively be deduced by an image approach due to Mr M. B. Glauert. To obtain the asymptotic disturbance at large distances, we replace the shear layer by a concentrated vortex sheet. Now, suppose first that the source of strength m is 'above' the vortex sheet, in the stream with velocity $V(\infty)$. By applying the condition of continuous pressure and flow direction across the vortex sheet, it may be shown that if squares of disturbances are neglected the flow in the upper region is equivalent to that due to the original source of strength m and a source of strength m_i (i for image) placed at its mirror-image in the vortex sheet; further, the flow in the lower region is that due to a source of different strength m_t (t for transmitted) at the position of the original source. Here,

$$m_i = \frac{V^2(-\infty)-V^2(\infty)}{V^2(-\infty)+V^2(\infty)}m, \qquad m_t = \frac{2V(\infty)V(-\infty)}{V^2(\infty)+V^2(-\infty)}m. \tag{85}$$

When the original source approaches close to the vortex sheet it almost coincides with its image, so effectively in the upper region we have a source of strength $m+m_i$, and in the lower region one of strength m_t. Finally, when the source is in the shear layer itself, at $y = 0$, the fluid produced by it fills a cylinder (stretching back from the source) of cross-sectional area $m/V(0)$; this measures how much the rest of the flow is pushed out. Its effect is the same as if a source of strength $mV(\infty)/V(0)$ were present just above the shear layer where the velocity is $V(\infty)$. Therefore, the effective source strengths m_1 and m_2 in the regions above and below the shear layer respectively should be

$$m_1 = (m + m_i)\frac{V(\infty)}{V(0)}, \qquad m_2 = m_i\frac{V(\infty)}{V(0)}, \tag{86}$$

which agree with (77) and (83).

The effective displacements c_1 and c_2 of the source (equations (78) and (84)) can probably not be calculated by such a method, as they are of the order of the thickness of the shear layer itself.

The limiting case when $V(-\infty)/V(\infty) \to 0$ is of interest; then both m_1 and m_2 tend to 0. This is connected with the fact that $m_i \to -m$ (the image of a source in a 'free streamline' separating a flow from a dead-air region being an equal sink), so that the source and its image cancel. They leave as the leading term in the disturbance in the upper region of uniform flow, a doublet whose strength

$$m_1 c_1 \to -\frac{m}{V(0)V(\infty)}\int_{-\infty}^{0} V^2(q)\,dq. \tag{87}$$

The case $V(y) = Ue^{\lambda y}$ studied in § 3 may be compared with this, since that distribution has in a sense $V(-\infty)/V(\infty) = 0$. But for large r (except where y and z are not large and x is positive) the disturbance in that case was found to fall off like $e^{-\lambda r}$, so that not only the effective source but also the effective dipole and higher multipole strengths are zero; possibly the $V(\infty)$ in the denominator in (87) might have led one to expect this.

Note that the net volume flow through the part of the surface of a large sphere, with centre at the origin, which lies outside the shear layer, is

$$\tfrac{1}{2}(m_1 + m_2) = \frac{V(\infty)V(-\infty)\{V(\infty) + V(-\infty)\}}{V(0)\{V^2(\infty) + V^2(-\infty)\}}\,m. \tag{88}$$

This differs from m, being less than m for a mixing region and greater for a wake. The balance is made up by flow across the part of the surface of the sphere immediately downstream of the source, where streamlines have been displaced. In a mixing region, for example, the displacement is towards the region of lower velocities, and so the velocity at any fixed point in space is increased. This point will be returned to in § 8.

We begin now to investigate the flow for large r within the shear layer, that is, where $|y|$ is not large and therefore equation (75) holds. This work depends on the result that

$$\int_0^\infty F(k)kJ_0(ks)\,dk \sim -\frac{F'(0)}{s^3} + \frac{3}{2}\frac{F'''(0)}{s^5} - \frac{3.5}{2.4}\frac{F^{v}(0)}{s^7} + \dots \tag{89}$$

as $s \to \infty$, if $F(k)$ is a bounded analytic function of k for $0 \leqslant k \leqslant \infty$. The author has been unable to find a clear-cut reference for this result, though it must presumably be well-known to people who frequently use the Hankel transform. It can be approached either by real- or complex-variable methods, the latter being simpler, though they assume rather more.

If, in fact, $F(k)$ is regular in a wedge $|\arg k| \leqslant \alpha$, then we write

$$J_0(ks) = \frac{K_0(-iks) - K_0(iks)}{\pi i} \tag{90}$$

(by Watson 1944, § 3.7 (8) and § 3.62 (5)), and divide the integral (89) into two parts, shifting the path of integration to $\arg k = +\alpha$ for the part involving $K_0(-iks)$ and to $\arg k = -\alpha$ for the part involving $K_0(iks)$. The integral becomes

$$\frac{1}{\pi i}\int_0^{\infty e^{i\alpha}} F(k)kK_0(-iks)\,dk - \frac{1}{\pi i}\int_0^{\infty e^{-i\alpha}} F(k)kK_0(iks)\,dk$$

$$= \frac{1}{\pi}\int_0^{\infty e^{i(\alpha-\frac{1}{2}\pi)}} F(it)itK_0(ts)\,dt + \frac{1}{\pi}\int_0^{\infty e^{i(\frac{1}{2}\pi-\alpha)}} F(-it)(-it)K_0(ts)\,dt. \tag{91}$$

Since $K_0(ts)$ falls off exponentially along each path in (91), the integrals may be asymptotically expanded, by a method directly analogous to the 'Borel' method used earlier in this section, as

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\frac{i^{n+1}F^{(n)}(0)}{n!}\int_0^{\infty e^{i(\alpha-\frac{1}{2}\pi)}} t^{n+1}K_0(ts)\,dt + \frac{(-i)^{n+1}F^{(n)}(0)}{n!}\times\right.$$

$$\left.\times\int_0^{\infty e^{i(\frac{1}{2}\pi-\alpha)}} t^{n+1}K_0(ts)\,dt\right]. \tag{92}$$

The contour integrals in (92) are clearly identical (and their values are simple combinations of factorials; see Watson 1944, § 13.21 (8)); hence the terms with n even in the sum vanish, and the rest take the form (89).

We wish to apply (89) to the case $F(k) = v_0(y, k)$, so the question of the analyticity as functions of k of the expressions (43) for v_0 arises. The analycitity of $v_1(y, k)$ and $v_2(y, k)$ and their derivatives with respect to y for $\mathscr{R}\{k\} \geqslant 0$ is easily proved by Picard's method from the integral equation (49). Hence v_0 will be analytic provided that $W(k)$ is non-zero. But the proof based on (48) that this is so can easily be extended to complex k with non-negative real part; in fact, if $W(k)$ vanished, so that a continuously differentiable solution of (26) tending to zero as $y \to \pm\infty$ existed, then the integral (48) would still be zero if k^2 were replaced by its real part and all other squares replaced by squares of moduli, and the conclusion that v_0 is identically zero would follow as before provided that $\mathscr{R}\{k^2\} \geqslant 0$. Hence we may take $\alpha = \frac{1}{4}\pi$ in the above discussion.

Equation (89), whose applicability to the case $F(k) = v_0(y, k)$ has thus been established, now gives with (25) and (75) that

$$v = \frac{mV(y)}{2\pi V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)}\left[\int_0^y \frac{dq}{Y(q)} - \frac{1}{Y(\infty)+Y(-\infty)} \times \right.$$
$$\left. \times \left\{\int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\,dq - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\,dq\right\}\right]\frac{1}{s^3} +$$
$$+O\left(\frac{1}{s^5}\right), \quad (93)$$

as $s = (x^2+z^2)^{1/2} \to \infty$ for values of y which are not large.

It is, perhaps, tiresome that the expression for v when r is large should take three different forms (79), (82) and (93), according as y is large and positive, large and negative, or not large at all, and it is natural to ask whether they can all be replaced by a single, uniformly valid asymptotic form. This does not seem to be possible. By careful inspection of (79) and (93) one can show that both are included in the more general expression

$$v = \frac{V(y)}{V(\infty)}\left\{-\left(m_1\frac{\partial}{\partial y} + m_1 c_1 \frac{\partial^2}{\partial y^2}\right)\frac{1}{4\pi r} - \frac{m_1}{4\pi r^3}\int_y^\infty \frac{Y(\infty)-Y(q)}{Y(q)}\,dq\right\} + O\left(\frac{1}{r^5}\right), \quad (94)$$

which therefore is valid both when y is large and positive and when $|y|$ is not large; but for large negative y (94) tends to

$$\frac{m_2 y}{4\pi r^3} - \frac{m_2 c_2}{4\pi r^3} \quad (95)$$

whereas by (82) there should be an additional term $3m_2 c_2 y^2/4\pi r^5$ for large negative y.

However, such an expression as (94) is just what one needs to integrate (15) (with lower limit $+\infty$) to obtain an expression for w when s but not $|y|$ is large. This gives simply

$$w = \frac{2Y(\infty)Y(-\infty)}{V(y)V(0)\{Y(\infty)+Y(-\infty)\}} \frac{mz}{4\pi s^3} + O\left(\frac{1}{s^4}\right), \quad (96)$$

whence, by (93) and the equation of continuity,

$$u = \frac{2Y(\infty)Y(-\infty)}{V(y)V(0)\{Y(\infty)+Y(-\infty)\}} \frac{mx}{4\pi s^3} - \frac{mV'(y)}{2\pi V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)} \times$$
$$\times \left[\int_0^y \frac{dq}{Y(q)} - \frac{1}{Y(\infty)+Y(-\infty)}\left\{\int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\,dq - \right.\right.$$
$$\left.\left. -\int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\,dq\right\}\right]\frac{1+x/s}{z^2} + O\left(\frac{1}{s^4}\right). \quad (97)$$

The leading terms of (96) and (97) show how the effective source strength varies *inversely* as the undisturbed flow velocity $V(y)$ within the shear layer, tending to m_1 (see (77)) as $y \to +\infty$ and to m_2 (see (83)) as $y \to -\infty$. This is connected with the fact that the pressure is constant across the shear layer far from the source, and that the pressure varies like $-\rho V(y)u$

along any streamline. The additional term in u, not inversely proportional to $V(y)$, is due to the displacement of the streamlines.

8. Theory for shear layers with small total variation of velocity

In § 4 to § 7 the theory has been developed for a general shear layer, and asymptotic expressions for the velocity field found, both as $r \to 0$ (in the form of terms of orders r^{-2} and r^{-1} plus an error of order 1), and as $r \to \infty$ (in the form of terms of order r^{-2} and r^{-3} plus an error of order r^{-4}). However, the detailed characteristics of the join between these solutions remain unknown, and it was to give some understanding of these that the solution to be described in this section was sought. This solution is valid for all values of x, y and z, but only for a restricted class of shear layers, namely those within which the total variation of the undisturbed velocity $V(y)$ is small compared with $V(y)$ itself. If the minimum value of $V(y)$ is $(1-\epsilon)$ times the maximum, we seek an approximate solution by consistently neglecting ϵ^2.

As before, we begin by obtaining $v_1(y,k)$. The integral equation (49) shows that, if the first power of ϵ be neglected, v_1 is simply e^{-ky}. This first approximation is valid uniformly, even for small k, since the fraction involving k within the integral is always less than $(z-y)$; and indeed the solution (60), valid in the limit $k \to 0$, differs from e^{-ky} only by a term of order ϵ.

Applying equation (49) again, we now see that

$$v_1(y,k) = e^{-ky}\left\{1+\int_y^\infty \frac{1-e^{-2k(q-y)}}{2k}\,\frac{V''(q)}{V_m}\,dq\right\}+O(\epsilon^2). \tag{98}$$

Equation (98) is similar to the earlier approximation (51), except that the exponential factor cannot now be suppressed because k is not being assumed large. However, the $V(q)$ in the denominator can be replaced by a mean value V_m (say, half the sum of the maximum and minimum of $V(y)$) with an error $O(\epsilon^2)$.

Similarly,

$$v_2(y,k) = e^{ky}\left\{1+\int_{-\infty}^y \frac{1-e^{2k(q-y)}}{2k}\,\frac{V''(q)}{V_m}\,dq\right\}+O(\epsilon^2), \tag{99}$$

and also

$$\left.\begin{aligned} v_1' &= e^{-ky}\left[-k-\frac{1}{2}\int_y^\infty\left\{1+e^{-2k(q-y)}\right\}\frac{V''(q)}{V_m}\,dq\right]+O(\epsilon^2),\\ v_2' &= e^{ky}\left[k+\frac{1}{2}\int_{-\infty}^y\left\{1+e^{2k(q-y)}\right\}\frac{V''(q)}{V_m}\,dq\right]+O(\epsilon^2),\end{aligned}\right\} \tag{100}$$

whence it follows easily that the Wronskian $v_1v_2'-v_2v_1'$ is simply

$$W(k) = 2k+\int_{-\infty}^\infty \frac{V''(q)}{V_m}\,dq+O(\epsilon^2) = 2k+O(\epsilon^2). \tag{101}$$

Next, by (46),

$$\left.\begin{aligned} A(k) &= \frac{\pi}{8\pi^2 k}\left\{k+\frac{1}{2}\int_{-\infty}^{0}(1+e^{2kq})\frac{V''(q)}{V_m}dq-\frac{V'(0)}{V_m}+O(\epsilon^2)\right\},\\ B(k) &= \frac{m}{8\pi^2 k}\left\{-k-\frac{1}{2}\int_{0}^{\infty}(1+e^{-2kq})\frac{V''(q)}{V_m}dq-\frac{V'(0)}{V_m}+O(\epsilon^2)\right\}.\end{aligned}\right\} \tag{102}$$

Hence, by (43), (98), (99) and (102), if ϵ^2 is neglected,

$$v_0(y,k) = \frac{m}{8\pi^2}\left\{e^{-k|y|}\operatorname{sgn}y-\frac{V'(0)+V'(y)}{2V_m}\frac{e^{-k|y|}}{k}-\frac{1}{2k}\times\right.$$
$$\left.\times\left(\int_y^{\infty}-\int_{-\infty}^{0}\right)e^{-k|2q-y|}\frac{V''(q)}{V_m}dq\right\}. \tag{103}$$

Equation (103) has been written in the form which gives clearest confirmation of the result proved in § 5, that v differs from the sum of the primary flow and the simple-shear secondary flow by an expression bounded as $r\to 0$. For it gives the Hankel transform (25), after the substitution $|2q-y| = 2l$ in the last integral, and use of the result (31), as

$$v = \frac{my}{4\pi r^3}-\frac{V'(0)+V'(y)}{2V_m}\frac{m}{4\pi r}-\frac{m}{8\pi V_m}\times$$
$$\times\int_{\frac{1}{2}|y|}^{\infty}\frac{V''(\frac{1}{2}y+l)-V''(\frac{1}{2}y-l)}{\sqrt{(4l^2+s^2)}}dl. \tag{104}$$

As $r\to 0$, this satisfies

$$v = \frac{my}{4\pi r^3}-\frac{V'(0)}{V_m}\frac{m}{4\pi r}-\left\{\frac{V''(0)y}{2V_m}\frac{m}{4\pi r}+\frac{m}{16\pi V_m}\int_0^{\infty}\frac{V''(l)-V''(-l)}{l}dl\right\}+o(1). \tag{105}$$

The terms in curly brackets in (105) may be called the 'tertiary flow' contribution to v; they are the limit as $r\to 0$ of what remains when the primary-flow and the simple-shear secondary-flow terms are taken away. The presence of the integral shows that this tertiary flow depends on the value of $V''(y)$ not only at the source itself but throughout the shear layer.

Expression (104) is not, however, the most suitable form of v for showing its behaviour away from the source. For this, we first integrate (103) by parts, substitute $2q = l$ and only then take its Hankel transform, namely

$$v = \frac{my}{4\pi r^3}+\frac{m}{4\pi}\left(\int_{2y}^{\infty}-\int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\frac{y-l}{\{(y-l)^2+s^2\}^{3/2}}dl. \tag{106}$$

(Alternatively, on integrating (104) by parts and changing the variable of integration we get (106).)

Equation (106) has a simple and valuable geometrical interpretation. It says that the y-component of disturbance velocity v is that due to the source of strength m at the origin, together with a line distribution of sources along certain stretches of the y-axis; to be precise, the stretches from $2y$ to ∞ and from $-\infty$ to 0 when y is positive, or $\left(\text{since } \int_{2y}^{\infty}-\int_{-\infty}^{0} \text{ is the same as } \int_0^{\infty}-\int_{-\infty}^{2y}\right)$ the stretches from 0 to ∞ and from $-\infty$ to $2y$ when y is negative. The source strength at the point $(0, l, 0)$ is $V'(\frac{1}{2}l)\,\mathrm{sgn}\, l/2V_m$ per unit length.

The values of w and u can be interpreted in the same way. Thus, by (15),

$$w = \frac{\partial}{\partial z}\left\{\frac{1}{V(y)}\int_{\infty}^{y} V(q)v(x,q,z)\,dq\right\}$$
$$= \frac{\partial}{\partial z}\int_{\infty}^{y} v(x,q,z)\,dq + \int_{\infty}^{y}\frac{V(q)-V(y)}{V_m}\,\frac{mq}{4\pi(q^2+s^2)^{3/2}}\,dq\Bigg\} \qquad (107)$$

if $O(\epsilon^2)$ be neglected, and when the integration is carried out we find that

$$w = \frac{mz}{4\pi r^3} + \frac{m}{4\pi}\left(\int_{2y}^{\infty} - \int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\,\frac{z}{\{(y-l)^2+s^2\}^{3/2}}\,dl, \qquad (108)$$

which is the z-velocity due to the distribution of sources described above. Also, on integrating

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \qquad (109)$$

from $-\infty$ to x to obtain u, we clearly get the x-velocity due to the same distribution of sources, except from one extra term in $\partial v/\partial y$ resulting from the dependence on y of one of the limits of integration in (106). This additional term on the right of (109) makes a contribution

$$-\int_{-\infty}^{x}\frac{m}{4\pi}\,\frac{V'(y)}{V_m}\,\frac{y}{(x^2+y^2+z^2)^{3/2}}\,dx \qquad (110)$$

to u, whence (by evaluating this integral)

$$u = \frac{mx}{4\pi r^3} + \frac{m}{4\pi}\left(\int_{2y}^{\infty} - \int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\,\frac{x}{\{(y-l)^2+s^2\}^{3/2}}\,dl - $$
$$-\frac{m}{4\pi}\,\frac{V'(y)}{V_m}\,\frac{y(1+x/r)}{y^2+z^2}. \qquad (111)$$

Thus, apart from the last term in (111), the complete system of disturbance velocities can be identified with the flow due to the source distribution along the y-axis described above.

This does not mean that it is irrotational. The velocity at a point (x, y, z) is that due, not to a fixed system of sources, but to one which itself varies with y. This gives additional terms in the y-derivatives of u and w which (together with the y- and z-derivatives of the extra term in u) prevent any component of the vorticity from vanishing. An actual expression for the vorticity is obtained later (equation (117), in which by (120) ϕ must be taken as $-m/4\pi r V_m$).

Mr M. B. Glauert has shown the author an ingenious derivation of the equivalent source distribution from his image theory mentioned in § 7. He regards the shear layer as made up of a very large number of elementary vortex sheets. The velocity at a point (x, y, z) due to the source at the origin is then represented as the velocity field of the original source itself, together with its images in all vortex sheets which do not pass ***between*** the origin and (x, y, z). This is permissible if $O(\epsilon^2)$ be neglected because (i) the 'transmitted' source strength m_t, by which the original source strength m needs to be replaced as a result of vortex sheets between the origin and (x, y, z), is equal to $m + O(\epsilon^2)$ (as (85) shows) because the difference of velocities across all these vortex sheets is $O(\epsilon)$, and (ii) the images of the image sources will have strength $O(\epsilon^2)$.

It follows that there will be image sources from $2y$ to ∞ and from $-\infty$ to 0 if $y > 0$, and from $-\infty$ to $2y$ and from 0 to ∞ if $y < 0$. Further, the source strength between $(0, l, 0)$ and $(0, l+dl, 0)$ due to the vortex sheet lying between $(0, \frac{1}{2}l, 0)$ and $(0, \frac{1}{2}l + \frac{1}{2}dl, 0)$, across which the velocity change is $V'(\frac{1}{2}l)\frac{1}{2}\,dl$, is seen from (85) to be $\{V'(\frac{1}{2}l)\operatorname{sgn} l/2V_m\}\,dl$. The source distribution is therefore exactly that obtained analytically.

To explain the additional term in (111), note that Glauert's vortex sheets are displaced by the source flow. To a first approximation (neglecting $O(\epsilon)$) the y-component of displacement is

$$\frac{1}{V_m}\int_{-\infty}^{x} \frac{my}{4\pi r^3}\,dx = \frac{my(1+x/r)}{4\pi V_m(y^2+z^2)}. \tag{112}$$

The disturbance velocity u will therefore have an additional component approximately

$$V\left\{y - \frac{my(1+x/r)}{4\pi V_m(y^2+z^2)}\right\} - V(y) \doteqdot -\frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{y(1+x/r)}{y^2+z^2}, \tag{113}$$

as in (111), with error $O(\epsilon^2)$.

The equivalent source distributions appropriate to a point with $y > 0$ and to one with $y < 0$ are illustrated for a mixing-region flow in figure 1. A general picture of the disturbance flows can be readily got from this figure. If y were actually greater than the greatest value it takes in the shear layer, the upper part of the equivalent source distribution would disappear altogether. In this region, therefore, the flow is the irrotational flow due to a fixed system of sources. Their total strength is easily verified to be m_1, and their total dipole moment $m_1 c_1$ (see § 7), to within an error $O(\epsilon^2)$. Similar results hold for values of y less than the least value it takes in the shear layer. Within the shear layers, the sources and sinks are so arranged as to produce a pronounced 'downwash' in the case of a mixing region, but this is somewhat mitigated in the case of a wake.

Far downstream of the source, where x is large and positive but y and z are not large, the disturbance velocities are dominated by the last term in (111) (interpreted in (113) as a displacement term); thus,

$$(u, v, w) \sim \left\{-\frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{2y}{y^2+z^2},\ 0,\ 0\right\}. \tag{114}$$

We may use (114) to check conservation of volume flow. The net volume flow, across that part of a large sphere (with centre the origin) which intersects the portion of shear layer downstream of the source where (114) holds, is

$$-\frac{m}{2\pi V_m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V'(y)\frac{y}{y^2+z^2}\,dydz = -\frac{m}{2V_m}\int_{-\infty}^{\infty} V'(y)\frac{y\,dy}{|y|}$$

$$= -\frac{m}{2V_m}\{V(\infty)+V(-\infty)-2V(0)\} = m - \tfrac{1}{2}(m_1+m_2), \tag{115}$$

where (88) has been used with $O(\epsilon^2)$ neglected. Together with the volume flow $\frac{1}{2}m_1$ across the part of the sphere above the shear layer, and $\frac{1}{2}m_2$ across the part below, this makes up exactly the rate m at which fluid is being produced at the source.

We here have exemplified from the case of small velocity spread how displacement of streamlines in the shear layer can remedy deficiencies between the rate of fluid output at the source and the rate of escape to regions far from the shear layer. In layers with large velocity spread the nature of the displacement is greatly altered (it includes secondary downwash and the like as well as the primary flow due to the source) but it must still produce this same effect. Of course, if terms of the order of the square of the source strength were included, the secondary trailing vorticity would be present, and then the streamlines far downstream would spiral about their mean positions; however, the conclusion would be unaltered.

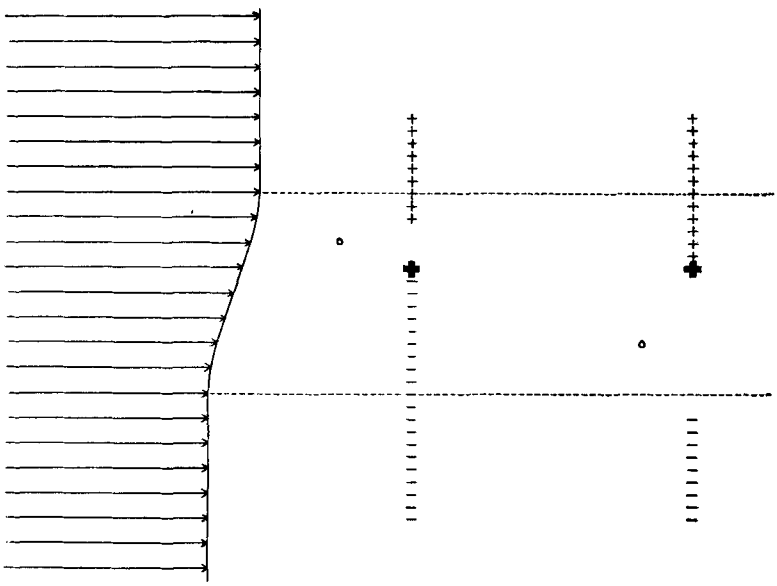

Figure 1. The equivalent source distribution, representing the effect that a weak source in a mixing region with small velocity spread makes at the point marked with a circle (which has $y > 0$ for the left-hand of the two distributions shown, and $y < 0$ for the right-hand one). The heavy plus sign represents the original source in each case, and the light plusses and minuses additional sources and sinks respectively. These can be regarded (Glauert) as images of the original source in the component vortex sheets of the shear layer.

9. Relation between the present solution and the exact-profile secondary flow

In the present paper the disturbance due to a weak source in a shear layer has been investigated by neglecting the square of the disturbance due to the presence of the source, no other approximation being made in § 2 to § 7. In § 8 the additional approximation of neglecting the square of ϵ (the relative spread of velocity in the shear layer) was made.

It is interesting to compare the results of the double approximation with those of another procedure, in which first the effect of a small non-uniformity in the oncoming stream on a basic irrotational flow (such as that due to the source) is considered by neglecting squares of ϵ, and the result is then further approximated by assuming small disturbances. This procedure was expounded briefly for the case of a general parallel flow in Lighthill (1956, p. 39), although the rest of that paper was devoted to the case of a uniformly sheared upstream flow.

The term 'secondary flow' may appropriately be used to describe any flow field computed by taking an irrotational flow with uniform upstream velocity as the first approximation (or primary flow), and going to a second approximation by allowing a small spread in the upstream velocity but neglecting its square. The secondary flow may be called a simple-shear secondary flow if the upstream flow is taken as uniformly sheared, and an exact-profile secondary flow if the true upstream velocity profile is used.

As shown in Lighthill (1956), secondary flows can be calculated by finding how the vortex lines of the upstream parallel flow become stretched and rotated by the primary flow. The secondary vorticity field having been found, some fairly arduous integrations are needed in general to determine the secondary velocities; but such integrations have been carried out for the simple-shear secondary flow about a sphere in Lighthill (1957 b).

The secondary vorticity field for any upstream velocity profile can be expressed in terms of the 'drift function' t for the primary flow. This function is such that planes of fluid initially at right angles to the stream are distorted by the primary flow into shapes given by equations t = constant; t represents the time at which a given fluid particle reaches a point, measured from when it would have reached $x = 0$ had the uniform upstream flow continued undisturbed. By considering how this distortion of planes of fluids affects the vortex lines of the secondary flow, which initially lie in such planes, one obtains the fairly simple equations (30) of Lighthill (1956) for the exact-profile secondary vorticity field.

These can be further simplified in the region away from the source of disturbance, where disturbances are themselves small, since in this region the drift function takes the simple form

$$t \sim \frac{x-\phi}{U}, \tag{116}$$

where ϕ is a 'disturbance potential', defined so that the full velocity potential of the primary flow is $U(x+\phi)$. The form of the secondary vorticity distribution in this region is easily deduced (Lighthill 1956, (32)) in terms of ϕ; in the special case when the upstream velocity is $V(y)$ (independent of z) it becomes

$$\operatorname{curl}(u,v,w) \sim \left\{ -V'(y)\frac{\partial\phi}{\partial z}, \quad -V'(y)\int_{-\infty}^{x}\frac{\partial^2\phi}{\partial y\,\partial z}\,dx, \right.$$
$$\left. V''(y)\int_{-\infty}^{x}\frac{\partial\phi}{\partial y}\,dx - V'(y)\int_{-\infty}^{x}\frac{\partial^2\phi}{\partial z^2}\,dx \right\}. \tag{117}$$

The disturbance velocity field associated with this disturbance vorticity field is

$$(u, v, w) = \left\{-V'(y)\int_{-\infty}^{x}\frac{\partial\phi}{\partial y}dx,\ V'(y)\phi,\ 0\right\} - \text{grad}\,\psi, \tag{118}$$

where

$$\nabla^2\psi = V''(y)\phi, \tag{119}$$

since the curl of (118) is (117) and its divergence vanishes (as was secured by subtracting grad ψ). In some cases it is necessary to add on to (118) an irrotational part representing the far field of the 'secondary trailing vorticity' (horseshoe vortices wrapped round the disturbance); this part is given in equation (85) of Lighthill (1957a).

When the disturbance is that due to a source, we may take

$$\phi = -\frac{m}{4\pi r U}. \tag{120}$$

The asymptotic form (118) of the secondary flow for large r then needs no addition due to the far field of the second trailing vorticity, because the latter is of smaller order, $O(r^{-2})$, as $r \to \infty$. Hence in particular

$$v \sim V'(y)\phi - \frac{\partial\psi}{\partial y} \tag{121}$$

as $r \to \infty$.

Now, the solution (121) for the small-disturbance form of the exact-profile secondary flow, which is its asymptotic form far from the source, can be identified with the solution of § 8, where the small disturbances due to a source are approximated for an upstream flow with small velocity spread. For, if $O(\epsilon^2)$ be neglected, equation (19) (except at the origin) becomes

$$\nabla^2 v = \frac{V''(y)}{V_m}\frac{my}{4\pi r^3}, \tag{122}$$

and it is a simple matter to verify that the solution (106) does indeed satisfy (122). Hence, if we were to write v in the form (121), where ϕ is given by (120), then ψ must satisfy

$$\frac{\partial}{\partial y}(\nabla^2\psi) = V'''(y)\phi + 2V''(y)\frac{\partial\phi}{\partial y} - \nabla^2 v \tag{123}$$

$$= V'''(y)\phi + V''(y)\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\{V''(y)\phi\}, \tag{124}$$

and integrating with respect to y we obtain equation (119) for ψ. Similarly, the identity of w and u in the two solutions may be checked.

However, though this shows that the solution of § 8 could be obtained by the alternative method leading to (118) and (119), the method adopted in this paper seems preferable, because it gives a lot of information about the solution to the more general problem in which ϵ^2 is not neglected, and because the direct solution of the Poisson equation (119) would in any case involve the same Fourier and other analysis which had to be introduced above.

10. Relation to theories of the displacement of the stagnation streamline

One interesting property of shear flow is that when a Pitot tube is placed in such a flow with the open end of the tube pointing upstream the pressure measured in the tube is the stagnation pressure not on the streamline approaching the tube directly along its axis but on one displaced by a measurable amount in the direction of higher velocities (Young & Maas 1936). This may be regarded as evidence of the presence of 'downwash' (flow at right angles to the undisturbed streamlines, in the direction of decreasing velocity) on the streamlines approaching the tube along its axis, and so it becomes a problem to compute this downwash from secondary-flow or other theories and calculate the displacement from it. This is the subject of a companion paper (Lighthill 1957 b), but there is one aspect of it which must be mentioned here, as it can be properly treated only by the present theory. This aspect is the behaviour of the downwash far ahead of the tube.

The 'downwash function' $D(s)$ is the value of $(-v)$ on the axis of the tube ($y = z = 0$) at $x = -s$, and is expected to be positive if $V'(0) > 0$. The most important region of downwash is near the tube; here $D(s)$ is best found from the simple-shear secondary flow, or from a higher approximation in a sequence starting from this. But the downwash so calculated falls off (see (57)) only as

$$D(s) \sim \frac{V'(0)}{V(0)} \frac{m}{4\pi s} \tag{125}$$

at points far enough ahead of the tube both for the disturbances to be small and for the tube to be representable by a source of strength

$$m = \tfrac{1}{4}\pi d_e^2 V(0), \tag{126}$$

where d_e is its external diameter. But the displacement calculated from (125) would be logarithmically infinite, since (125) is not integrable up to $s = \infty$. If tertiary-flow terms were included, they would be even greater for large s, and still worse infinities would arise in the displacement.

The difficulty can be resolved by using the complete solution of this paper in the region where s is large enough for the disturbance to be small and for the tube to be representable by a source of strength (126). In this region the downwash falls off only initially like (125) (see (57)), and later like s^{-3}; more precisely, by (93), we have

$$D(s) \sim \frac{m}{2\pi s^3} \frac{Y(\infty)Y(-\infty)}{\{Y(\infty)+Y(-\infty)\}^2} \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\,dq - \right.$$
$$\left. - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\,dq \right\} \tag{127}$$

as $s \to \infty$. The difficulty of $D(s)$ not being integrable up to $s = \infty$ then disappears.

The problem is thus treated by means of two overlapping solutions. For small to moderate r, the simple-shear secondary-flow theory is used; for moderate to large r the exact-profile small-disturbance theory is used.

Expression (125) is valid in the overlap range of 'moderate' r; for it represents the asymptotic behaviour of the simple-shear secondary-flow theory as $r \to \infty$ and also that of the exact-profile small-disturbance theory as $r \to 0$.

To apply this to the calculation of displacement, let s_0 be a value of s in this overlap region. Then the displacement of the dividing streamline due to downwash $D(s)$ for $s < s_0$ is calculated from the simple-shear secondary flow. For $s > s_0$ the disturbances are small, and the contribution to the displacement from downwash in this region is

$$\frac{1}{V(0)} \int_{s_0}^{\infty} D(s)\, ds, \tag{128}$$

where $D(s)$ is given by the theory of this paper.

Now, we can write the contribution (128) as

$$\frac{1}{V(0)} \int_{s_0}^{s_c} \frac{V'(0)}{V(0)} \frac{m}{4\pi s} ds, \tag{129}$$

provided that

$$\frac{V'(0)}{V(0)} \frac{m}{4\pi} \log s_c = \lim_{s_c \to \infty} \left\{ \int_{s_0}^{\infty} D(s)\, ds + \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log s_0 \right\}, \tag{130}$$

and in the limit on the right the downwash function $D(s)$ is taken to be that given by the small-disturbance theory right down to $s_0 = 0$. It is because *this* $D(s)$ is asymptotic to (125) as $s \to 0$ that no substantial difference is made by allowing s_0 to become smaller and vanish in (130), since s_0 has already been assumed sufficiently small for the approximation (125) to be good.

We call s_c, defined by (130), the upstream cut-off of the simple-shear secondary flow. The displacement for $s > s_0$, as (129) shows, is equal to what it would be if the downwash were equal to the simple-shear secondary-flow value (125) up to $s = s_c$ and zero for $s > s_c$—in other words, if the secondary flow were cut off at $s = s_c$.

A precise expression for the cut-off value s_c can be obtained if the theory of § 8 for the case of a small velocity spread be used. Then, by (104),

$$D(s) = \frac{V'(0)}{V(0)} \frac{m}{4\pi s} + \frac{m}{8\pi V_m} \int_0^{\infty} \frac{V''(l) - V''(-l)}{\sqrt{(4l^2 + s^2)}} dl. \tag{131}$$

Hence

$$\begin{aligned} \int_{s_0}^{\infty} D(s)\, ds &= \lim_{s_1 \to \infty} \int_{s_0}^{s_1} D(s)\, ds \\ &= \lim_{s_1 \to \infty} \left\{ \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log \frac{s_1}{s_0} + \frac{m}{8\pi} \int_0^{\infty} \frac{V''(l) - V''(-l)}{V_m} \times \right. \\ &\qquad \left. \times \left(\sinh^{-1} \frac{s_1}{2l} - \sinh^{-1} \frac{s_0}{2l} \right) dl \right\} \\ &= \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log \frac{1}{s_0} + \frac{m}{8\pi} \int_0^{\infty} \frac{V''(l) - V''(-l)}{V_m} \left(\log \frac{1}{l} - \sinh^{-1} \frac{s_0}{2l} \right) dl, \end{aligned} \tag{132}$$

whence, by (130),

$$\log s_c = \frac{1}{2V'(0)} \int_0^\infty \{-V''(l) + V''(-l)\} \log l \, dl. \qquad (133)$$

Since the denominator $2V'(0)$ in (133) is the value of the integral if the $\log l$ in it be suppressed, we can say that $\log s_c$ is the average value of $\log l$ in the shear layer, if the 'weight' used in averaging is $-V''(l)\operatorname{sgn} l$. This makes s_c itself a 'geometric' mean of l in the shearing layer. (Here, l means the difference in y-coordinate between a general point and the source.) Thus, s_c is of the order of the layer width, as indeed is easy to see geometrically from the 'equivalent source distribution' described in § 8. In more general problems it is reasonable to suppose that s_c is still of the order of the layer width.

We may consider also the implications of the present theory for problems like the displacement effect of a sphere, as treated by Hall (1956) and by Lighthill (1957 b) on the assumption that the inviscid flow is a good enough approximation (in other words, that separation and wake flow do not influence the displacement). The upstream effect of the sphere is that of a doublet of strength

$$m = 2\pi a^3 V(0), \qquad (134)$$

if the sphere has radius a and centre the origin. With this value of m, the flow in the region of small disturbances (due to the doublet) is given simply by taking the x-derivative of the velocity field of this paper.

The upstream cut-off of the simple-shear secondary flow is not crucial in this case, since the simple-shear secondary downwash is itself integrable up to $s = \infty$. However, the cut-off reduces the displacement below that obtained by this integration, and the reduction can be calculated, as follows.

If $D_s(s)$ is the downwash function due to the source studied hitherto in this paper, and $D_d(s) = -D_s'(s)$ is that due to the doublet, then the displacement for $s > s_0$ is

$$\frac{1}{V(0)} \int_{s_0}^\infty D_d(s)\, ds = \frac{1}{V(0)} D_s(s_0)$$
$$\doteqdot \frac{1}{V(0)} \left\{ \frac{V'(0)}{V_m} \frac{m}{4\pi s_0} + \frac{m}{16\pi V_m} \int_0^\infty \frac{V''(l) - V''(-l)}{l} dl \right\}, \qquad (135)$$

where (105) has been used to approximate to $D_s(s_0)$. Now, the first term in curly brackets is the simple-shear secondary-flow value, and so the difference, between the true displacement and that given by assuming that the simple-shear secondary flow persists to infinity, is given approximately (using the value (134) for m) by

$$\frac{a^3}{8V_m} \int_0^\infty \frac{V''(l) - V''(-l)}{l} dl. \qquad (136)$$

This is presumably negative when $V'(0) > 0$, since the numerator of the integrand is then negative on the average. Thus the correction (136) due to the cut-off is a reduction, as expected.

Actually, it was shown in Lighthill (1957 a) that as well as the doublet of strength (134) the upstream influence of the sphere includes also a rather smaller contribution due to the 'secondary trailing vorticity'. This is equivalent to the effect of a line of doublets with axis the y-axis stretched all along the positive x-axis. However, it may be shown that the displacement resulting from this line of doublets is estimated correctly even if the cut-off studied above is neglected, and hence the validity of the correction (136) remains unaffected. For the practical significance of this correction, see Lighthill (1957 b).

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